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Polarization Tensor: Between Biology and Engineering

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ABSTRACT

There has been a lot of interest over recent years in the study of mathematical aspects and applications of the polarization tensor. This promising terminology appears widely in electric and electromagnetic inverse problems. Our main purpose in this paper then is to review the polarization tensor biologically in electro-sensing by a weakly electric fish and in the engineering problems which are based on the Eddy current principle. Here, the mathematical formulations of the polarization tensor for both cases are firstly presented. At the same time, a few related applications will also be briefly explained.

Keywords: boundary integral equations, conductivity, Eddy currents, electro-sensing fish, matrices, metal detectors, permeability.

1. Introduction

Due to numerous efforts in studying it lately, the roles of the polarization tensor (PT) in electric and electromagnetic applications are becoming wider. In electrical imaging, instead of reconstructing back the image of a small conducting object (Ammari and Kang, 2007), the PT can be fitted to describe objects such as during electro-sensing by a weakly electric fish in Taufiq and Lionheart, 2012, as it offers lower computational cost. Meanwhile, because of several advantages for example as mentioned in Blitz, 1991, the Eddy current principle is implemented in metal detector to locate and characterize metallic objects. The terminology PT is then adapted here in order to identity several other properties of the targeted metal such as shape, orientation and object material (see Marsh *et al*., 2013; Dekdouk *et al*., 2013, and Marsh *et al*., 2014). As the study of PT is quickly developed, the main purpose of this paper is to review past and present development about the PT especially in both applications with the hope to stimulate future studies about them.

The concept and mathematical aspects of the generalized polarization tensor (GPT) was properly documented in Ammari and Kang, 2007 and applied to the Electrical Impedance Tomography (EIT) system through "Calderon's Inverse Problem" formula i.e. the inverse conductivity equation of the electric potential in a conductive body. Depending wholly on the object, there were two ways to determine its PT at specified value of conductivity according to Ammari and Kang, 2007, which were through an integral operator or by solving a transmission problem of a partial differential equation (PDE). Both approaches however were only possible by numerical method.

Because of some similarities between the EIT system and electro-sensing by a weakly electric fish (Nelson, 2009), the first order PT of the GPT was investigated by Taufiq and Lionheart, 2012, to see its role biologically in electro-sensing. Here, the first order PT for several objects in the experiment conducted by von der Emde and Fetz, 2007, were computed and compared. On the other hand, the theory of PT for Eddy current approximation to Maxwell's equations was recently introduced by Ammari *et al.*, 2013, where based on their studies, the PT for a specified object in the problem could so far be determined by solving a transmission problem of another PDE. A method to identify an arbitrary shaped target in the metal detector by using the PT was also introduced in Ammari *et al*., 2014. This study was then further extended and explored by Ledger and Lionheart, 2013, in order to apply it to the related engineering applications.

In order to further discuss about these areas of the PT, this paper will be organized as follows. The next section will review briefly about the GPT. Section 3 then explains the terminology PT in electro-sensing by weakly electric fish. After that, we will proceed to the PT for the Eddy current problem in Section 4. Finally, the last section will summarize and conclude the study.

2. Generalized Polarization Tensor

We firstly consider here the PT that originates from a transmission problem of inverse conductivity equation which has being discussed by many literatures. Consider a Lipschitz bounded domain *B* in $\mathbb{R}^3$ such that the origin *O* is in *B*. Let the conductivity of *B* be equal to *k* where $0 < k \neq 1 < +\infty$. Suppose that *H* is a harmonic function in $\mathbb{R}^3$ and *u* is the solution to the following problem

$$\begin{cases} \operatorname{div}(1 + (k-1)\chi(B)\operatorname{grad}(u)) = 0 & \text{in} \quad \mathbb{R}^3 \\ u(x) - H(x) = O(|x|^{-2}) & \text{as} \quad |x| \to \infty \end{cases} \tag{1}$$

where χ denotes the characteristic function of *B*. The mathematical formulation (1) actually appears in many industrial applications such as medical imaging of EIT system, landmine detector and material sciences (Holder , 2005; Ammari and Kang, 2007; Adler *et al.*, 2011).

The PT is then defined by Ammari and Kang, 2007, through the following far-field expansion.

$$(u-H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!\,j!} \partial_x^i \Gamma(x) M_{ij}(k,B) \partial^j H(0) \text{ as } |x| \to +\infty \tag{2}$$

for *i*, *j* multi indices, Γ is the fundamental solution of the Laplacian and $M_{ij}(k,B)$ is the generalized polarization tensor (GPT). The GPT is usually referred as the dipole in electromagnetic applications by physicists because it shows the distribution of the conductivity of the object.

Furthermore, Ammari and Kang, 2007, extends the definition of GPT in (2) through an integral equation over the boundary of *B* by

$$M_{ij}(k,B) = \int_{\partial B} y^j \phi_i(y) d\sigma(y) \tag{3}$$

where $\phi_i(y)$ is given by

$$\phi_i(y) = \left(\lambda I - K_B^*\right)^{-1} \left(\nu_x \cdot \nabla x^i\right)(y) \tag{4}$$

for $x, y \in \partial B$ with identity *I* and ν_x is the outer unit normal vector to the boundary ∂B at *x*. Here, λ is defined by $\lambda = (k+1)/2(k-1)$ and K_B^* is a singular integral operator defined with Cauchy principal value *P.V.* by

$$\kappa_B^* \phi(x) = \frac{1}{4\pi} \text{p.v.} \int_{\partial B} \frac{(x-y)\cdot \nu_x}{|x-y|^3} \phi(y) d\sigma(y) \,. \tag{5}$$

Consequently, the PT of B at any conductivity k, $0 < k \neq 1 < +\infty$ can be directly determined if B and its conductivity are known as given by (3), (4) and (5). It does not depend on the position of B and can be obtained without solving (2). Furthermore, it is shown in Ammari and Kang, 2007, that the PT rotates as B rotates so the PT also depends on the orientation of B. If B is an ellipsoid, an analytical formula for its first order PT as 3×3 matrix is also given in Ammari and Kang, 2007.

Now, a transmission problem with the following equations is considered

$$
\begin{aligned}
&\Delta\psi_i(x) = 0, x \in B \cup (\mathbb{R}^3 \setminus \bar{B}) \\
&\psi_i |_+ (x) - \psi_i |_- (x) = 0, \quad x \in \partial B \\
&\left.\frac{\partial \psi_i}{\partial \nu}\right|_+ (x) - k \left.\frac{\partial \psi_i}{\partial \nu}\right|_+ (x) = \nu.\nabla x^j, \quad x \in \partial B \\
&\psi_i(x) \to 0 \quad \text{as} \quad |x| \to +\infty \quad \text{if} \quad d = 3,
\end{aligned} \tag{6}
$$

where ψ_i also satisfies the following decay estimates at infinity

$$\psi_i(x) - \Gamma(x) \int_{\partial B} \nu \cdot \nabla y^i d\sigma(y) = O\left(|x|^{-2}\right) \text{ as } |x| \to +\infty .$$

By finding the unique solution ψ_i to the problem (6), Ammari and Kang, 2007, proved that

$$\left(\lambda I - K_B^*\right)\left(\left.\frac{\partial \psi_i}{\partial \nu}\right|_-\right)(x) = \frac{1}{k-1}\left(-\frac{1}{2} I + K_B^*\right)(\nu \cdot \nabla y^i)(x) \tag{7}$$

for $x \in \partial B$. The PT can then be alternatively determined from

$$M_{ij}(k, B) = (k-1)\int_{\partial B} x^j \frac{\partial x^i}{\partial \nu} d\sigma(x) + (k-1)^2 \int_{\partial B} x^j \left.\frac{\partial \psi_i}{\partial \nu}\right|_- (x) d\sigma(x). \tag{8}$$

In their study, Ammari and Kang, 2007, specifically applied and related the PT to the theory of dilute composite material, electrical impedance and elastic imaging. As computational aspects of the GPT were not focused there, a method to compute the GPT for two dimensional domains was developed in (Capdeboscq *et al.*, 2011). For our purpose, we had discussed the procedure to determine the first order PT for three dimensional domains in Taufiq and Lionheart, 2013a. We then improved the results by solving (3), (4) and (5) with a software called as *BEM++* as presented in Taufiq and Lionheart, 2013b. This method was then applied in our further study on the role of the first order PT during electro-sensing by weakly electric fish.

3. The First Order GPT in Electro-Sensing Fish

A weakly electric fish in the rivers of South America and Africa performs electro-sensing in order to navigate as well as to characterize objects and locate prey (von der Emde, 2007). The fish is normally equipped with a single electric organ to discharge electric and has hundred of voltage sensing cells on the surface of its body. Since the fish typically moving through the water during electro-sensing, its single electric source looks to act in a similar way as switching between driven electrodes in an EIT system (Nelson, 2009). Because of this, we believe similar approach from the EIT can be used to study the fish. Assuming that the fish do not perform complete image reconstruction in the real time due to the complexity of the problem, the other way for the fish to electrically recognize the object is possibly by fitting the first order GPT for the object.

Following Taufiq and Lionheart, 2012, let the electrical conductivity in the region exterior to a weakly electric fish be σ and suppose that there is an isolated object *B* which is assumed to be a Lipschitz bounded domain in $\mathbb{R}^3$ at some distance from the fish. Consider the domain $\Omega = \mathbb{R}^3 - F$ where *F* is the fish and for any point $x \in \mathbb{R}^3$,

$$\sigma(x) = \begin{cases} 1 \ , & x \in \Omega - B \\ k \ , & x \in B \end{cases}$$

where *k* constant. According to Ammari and Kang, 2007, if *u* is the voltage in the region Ω then the perturbation in the voltage due to a small object *B* can be approximated by an asymptotic expansion where the dominant term of the expansion is determined by the Polarization Tensor (PT).

Now, if a harmonic function *H* is the voltage in Ω without the object *B* then from (2),

$$(u - H)(x) = -\nabla\Gamma(x) \cdot M\nabla H(O) + O(1/|x|^{-2}) \qquad (9)$$

where the origin $O \in B$, $\Gamma(x) = -(4\pi |x|)^{-1}$ and *M* is the first order PT for *B*. Here, *M* for an object *B* at conductivity *k* is a real symmetric 3×3 matrix and is obtained by setting $i, j = (1,0,0),(0,1,0)$ and $(0,0,1)$ in (3) or (8) which is in the form

$$M(k,B)=\begin{bmatrix} M_{(1,0,0)(1,0,0)} & M_{(1,0,0)(0,1,0)} & M_{(1,0,0)(0,0,1)} \\ M_{(0,1,0)(1,0,0)} & M_{(0,1,0)(0,1,0)} & M_{(0,1,0)(0,0,1)} \\ M_{(0,0,1)(1,0,0)} & M_{(0,0,1)(0,1,0)} & M_{(0,0,1)(0,0,1)} \end{bmatrix}. \tag{10}$$

Since B is described by (9), M can then be used instead of full $(u-H)(x)$ to describe B.

In order to examine the role of the PT in electro-sensing, we initiated an investigation on several experiments conducted by von der Emde and Fetz, 2007, about the abilities of elephantnose fish *Gnathonemus petersiis* to distinguish different objects through electrolocation (a type of electro-sensing). At this stage, only the first order PT for a few objects used in the experiment were considered and calculated by using our method in Taufiq and Lionheart, 2013a and Taufiq and Lionheart, 2013b, where two objects were said to be electrically similar if their first order PT were the same. We then reanalyzed their findings based on those the PT to look for any evidences that may support our hypothesis that the fish used the first order PT as part of its object recognition algorithm.

Figure 1: A petersii elephantnose fish

When the fish were trained in von der Emde and Fetz, 2007, to accept and reject two different objects, we found in Taufiq and Lionheart, 2012, that the fish needed longer time to accept and reject two objects if the difference between PT for both objects was small. In addition, the study conducted in Taufiq and Lionheart, 2013c, suggested that the same fish could measure the difference between the first order PT for two objects with different sizes or type of materials before making decision about the objects. Basically, after the fish were able to correctly choose what they were trained to accept and reject, they would most likely choose objects according to the way they were trained (von der Emde and Fetz, 2007).

Recently, we also learned in Taufiq and Lionheart, 2014, that in general, given the object that it was trained to accept and a few new objects, the percentage for the fish to choose what it was trained to accept increased as the difference between the first order PT for the object that it was trained to accept and the new object given to it increased. The fish would easily accept what it was trained to accept if given to it the object it was trained to accept and a new object where the difference of the PT for both objects was large. On the other hand, the fish would easily accepted any new object if given to them the object it was trained to reject and the new object regardless the difference between the first order PT for both objects. This behavior probably caused by strong influence during the training to reject the object. All these results were consistent with our hypothesis that the first order PT had some roles in electro-sensing of the fish.

We are currently proposing a few potential studies that can be conducted to further justify the role of the first order PT during electro-sensing in our next publication. One possible experiment that can be conducted is to ask the fish to distinguish two objects that has the same first order PT. We have presented a technique to construct two objects that have the same first order PT in Taufiq and Lionheart, 2013d, to achieve this. In the future, if we found that the fish can discriminate two objects with the same first order PT, we can then say it uses more than the first order PT in their recognition mechanism. It might use the higher order PT as well.

4. Polarization Tensor for Eddy Current Problem

Metal detection is highly regarded in industrial and engineering applications. Therefore, improving metal detectors are very essential for examples to increase the correct alarms during security screening and provide better safety when removing land mines. As metal detector is built according to Eddy current principles, recent study in electromagnetic suggests that one possible approach is to locate and characterize the target by using the PT of the Eddy current. In their engineering works, the PT for Eddy current was implemented to a metal detector prototype for security screening by Marsh *et al*., 2013, and Marsh *et al*., 2014, where it described location, dimension, orientation and material property of the potential threat objects. On the other hand, Dekdouk *et al.*, 2012, conducted several experiments by adapting the PT to a different type of metal detector in order to identify buried metallic landmines in the contaminated environmental fields.

In general, metal detectors in the previous experiments had a standard transmitting and receiving coil for field measurements. The PT, M of the target in the metal detector was then derived from the relation

$$V_{ind} = H^T M H^R \qquad (11)$$

where V_{ind} was the induced eddy current field due to the presence of the target at some location while both H^T and H^R were the fields generated by the transmitting and receiving coil respectively. By assuming it exists, M was computed by numerical optimization technique after V_{ind}, H^T and H^R were measured in each study. Thus, M for the tested target was actually reconstructed based on (11) and was only an approximation. It was not computed according to a specific formula based on the target as such formula was not yet available at that time.

As our information about what to detect increases from previous experiences, the formula of M if exists will enable us to accurately compute the PT and hence improving its implementation to the metal detector. The formula of the PT in this case is only recent and firstly derived by Ammari *et al*. (2013). In this case, the PT describes the perturbation in the magnetic fields generated by the Eddy current due to the presence of an object. Thus, the aim of this section is to review mathematical background of this PT as well as to discuss a few ongoing studies and applications about it. However, mathematical formulation of the Eddy current itself is not discussed but is referred from (Rodriguez and Valli (2010)).

(a) (b)

Figure 2: Metal detectors in (a) Airport security scanning and (b) Land-mine detector

Suppose that there is an object of the form $B_\alpha = z + \alpha B$ included in $\mathbb{R}^3$ which means that the object can be expressed in terms of smooth and bounded unit domain $B \in \mathbb{R}^3$ placed at the origin, scaled by the object size α as well as translated by the vector z. Introduce

$$\mu_\alpha = \begin{cases} \mu_* & \text{in} \quad B_\alpha \\ \mu_0 & \text{in} \quad \mathbb{R}^3 \setminus B_\alpha \end{cases}, \quad \sigma_\alpha = \begin{cases} \sigma_* & \text{in} \quad B_\alpha \\ 0 & \text{in} \quad \mathbb{R}^3 \setminus B_\alpha \end{cases}$$

where μ_0 denotes the magnetic permeability of the free space $\mathbb{R}^3$ while both μ_* and σ_* denote the permeability and conductivity of the object. Here, μ_* and σ_* are assumed to be constant.

Let $\boldsymbol{E}_\alpha$ and $\boldsymbol{H}_\alpha$ be the time harmonic Eddy current fields (electric and magnetic) in the presence of B_α that result from a current source $\boldsymbol{J}_0$ located outside B_α. By assuming $\nabla \cdot \boldsymbol{J}_0 = 0$ in $\mathbb{R}^3$, both fields $\boldsymbol{E}_\alpha$ and $\boldsymbol{H}_\alpha$ satisfy the Eddy current equations

$$\begin{gathered}\nabla \times \boldsymbol{E}_\alpha = \mathrm{i}\omega\mu_\alpha \boldsymbol{H}_\alpha \quad \text{in} \quad \mathbb{R}^3, \\ \nabla \times \boldsymbol{H}_\alpha = \sigma_\alpha \boldsymbol{E}_\alpha + \boldsymbol{J}_0 \quad \text{in} \quad \mathbb{R}^3, \\ \boldsymbol{E}_\alpha(x) = O(|x|^{-1}),\ \boldsymbol{H}_\alpha(x) = O(|x|^{-1}) \ \text{as} \ |x| \to \infty\end{gathered}$$

where i is the standard imaginary unit and ω is the angular frequency from the current source. On the other hand, without the object B_α, the fields $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ that result from time varying current source satisfy

$$\begin{gathered}\nabla \times \boldsymbol{E}_0 = \mathrm{i}\omega\mu_0 \boldsymbol{H}_0 \quad \text{in} \quad \mathbb{R}^3, \\ \nabla \times \boldsymbol{H}_0 = \boldsymbol{J}_0 \quad \text{in} \quad \mathbb{R}^3 \\ \boldsymbol{E}_0(x) = O(|x|^{-1}),\ \boldsymbol{H}_0(x) = O(|x|^{-1}) \ \text{as} \ |x| \to \infty .\end{gathered}$$

By letting $\nu = \omega\mu_0\sigma_*\alpha^2$, Ammari *et al.* (2013) derived the perturbation of magnetic field $\boldsymbol{H}_\alpha(\boldsymbol{x}) - \boldsymbol{H}_0(\boldsymbol{x})$ at position $\boldsymbol{x}$ and away from z due to the presence of B_α as follows

$$\begin{aligned}\left(\boldsymbol{H}_\alpha - \boldsymbol{H}_0\right)(\boldsymbol{x}) = -\frac{\mathrm{i}\nu\alpha^3}{2}\sum_{i=1}^{3}\boldsymbol{H}_0(\boldsymbol{z})_i \int_B \boldsymbol{D}_{\boldsymbol{x}}^2 G(\boldsymbol{x},\boldsymbol{z})\boldsymbol{\xi} \times (\boldsymbol{\theta}_i + \boldsymbol{e}_i \times \boldsymbol{\xi}) d\boldsymbol{\xi} + \ldots \\ \ldots\ \alpha^3\left(1 - \frac{\mu_0}{\mu_*}\right)\sum_{i=1}^{3}\boldsymbol{H}_0(\boldsymbol{z})_i \boldsymbol{D}_{\boldsymbol{x}}^2 G(\boldsymbol{x},\boldsymbol{z})\int_B \left(\boldsymbol{e}_i + \frac{1}{2}\nabla \times \boldsymbol{\theta}_i\right) d\boldsymbol{\xi} + \boldsymbol{R}(\boldsymbol{x}) \qquad (12)\end{aligned}$$

when $\alpha \to 0$ and $\nu = O(1)$ for every $\boldsymbol{\xi} \in \boldsymbol{B}$. $G(\boldsymbol{x},\boldsymbol{z}) = \dfrac{1}{4\pi|\boldsymbol{x} - \boldsymbol{z}|}$ is the free space Laplace Greens function and $\boldsymbol{R}(\boldsymbol{x}) = O(\alpha^4)$ is a small remainder term. Furthermore, for $i = 1,2,3$, $\boldsymbol{e}_i$ is a unit vector for the ith Cartesian coordinate direction, $\boldsymbol{H}_0(\boldsymbol{z})_i$ denotes the ith element of $\boldsymbol{H}_0(\boldsymbol{z})_i$ and $\boldsymbol{\theta}_i$ is the solution to the transmission problem

$$\begin{aligned} \nabla_\xi \times \mu^{-1}\nabla_\xi \times \boldsymbol{\theta}_i - \mathrm{i}\omega\sigma\alpha^2\boldsymbol{\theta}_i = \mathrm{i}\omega\sigma\alpha^2\boldsymbol{e}_i \times \boldsymbol{\xi} \ \text{ in } \ B \cup B^c, \\ \nabla_\xi \cdot \boldsymbol{\theta}_i = 0 \ \text{ in } \ B^c, \\ \left[\boldsymbol{\theta}_i \times \boldsymbol{n}\right]_\Gamma = 0 \ \text{ on } \ \Gamma, \\ \left[\mu^{-1}\nabla_\xi \times \boldsymbol{\theta}_i \times \boldsymbol{n}\right]_\Gamma = -2\left[\mu^{-1}\right]_\Gamma \boldsymbol{e}_i \times \boldsymbol{n} \ \text{ on } \ \Gamma, \\ \boldsymbol{\theta}_i(\boldsymbol{\xi}) = O(|\boldsymbol{\xi}|^{-1}) \ \text{ as } \ |\boldsymbol{\xi}| \to \infty \end{aligned} \tag{13}$$

where $\boldsymbol{n}$ is the outward normal vector to the boundary of unit domain B denoted by Γ while σ and μ represents the conductivity and permeability of B.

From this introductory, (12) is expressed in the alternative compact form by Ledger and Lionheart (2013) as

$$\left(\boldsymbol{H}_\alpha - \boldsymbol{H}_0\right)(\boldsymbol{x}) = \boldsymbol{D}_{\boldsymbol{x}}^2 G(\boldsymbol{x},\boldsymbol{z}) M \boldsymbol{H}_0(\boldsymbol{z}) + \boldsymbol{R}(\boldsymbol{x}) \tag{14}$$

where M is the desired PT and can be regarded as a rank 2 or a rank 4 tensor. M is built by Ledger and Lionheart, 2013, from the combination of the original permeability polarization tensor, **n** and the conductivity polarization tensor, **c** of Ammari *et al.*, 2013, given by

$$\mathbf{c} = \frac{1}{2}\boldsymbol{e}_l \times \int_B \xi_{l'}(\boldsymbol{\theta}_i + \boldsymbol{e}_i \times \boldsymbol{\xi})\,d\boldsymbol{\xi}, \tag{15}$$

$$\mathbf{n} = \left(1 - \frac{\mu_0}{\mu_*}\right)\int_B \left(\boldsymbol{e}_i + \frac{1}{2}\nabla \times \boldsymbol{\theta}_i\right) d\boldsymbol{\xi},$$

for $l, l' = 1,2,3$. The engineering prediction about the polarization tensor in (11) was then confirmed by treating M as the rank 2 tensor. Several properties of M as the rank 4 and the rank 2 tensor were also investigated later on. Moreover, a *hp*-Finite Element method to compute M for both forms was also introduced in Ledger and Lionheart, 2013, to describe M for a few common objects.

Based on these derivations, the PT for a known object in the related applications can now be determined by using the established formulas (13) and (15). Our next study will focus on computing and describing a few objects which appear in the metal detector such as coin, knife, gun, belt buckle and land mines by using their PT. It is also our aim to compare between the reconstructed PT and the computed PT with the hope to improve metal detection by the PT in the future.

5. Discussions and Conclusions

During this study, the mathematical formulation of the first order PT in electro-sensing fish which is based on the first order GPT is highlighted. In order to achieve this, the mathematical formulation and a few studies about the GPT itself are firstly presented. We then discuss previous studies about the role of the first order PT in this area as well as current and future planned investigation to further justify it. On the other hand, a new mathematical formulation of the PT for Eddy current problems which is developed to improve metal detectors in engineering applications is also reviewed here. While the PT in electro-sensing is defined in the perturbated electrical fields, the PT for Eddy current is introduced based on the perturbation in the magnetic field of the induced current. For each case, the perturbation to the field is caused due to the presence of an object. However, only electrical conductivity of the object matters for the PT in electro-sensing but both conductivity and permeability of the object must be considered in the PT for eddy current. By relating these two PT, we believe that a few good strategies by weakly electric fish to detect and describe objects can be modified and adapted in metal detector through the PT in the future.

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